[7]Atassi, H. M. and Gebert, G. A., "Modification of Turbulent Boundary Layer Structure by Large-Eddy Breakup Devices," *Turbulent Drag Reduction by Passive Means*, Royal Aeronautical Society, London, England, 1987, Palo Alto, CA, pp. 432–456.

[8]Sears, W. R., "Some Aspects of Non-Stationary Airfoil Theory and Its Practical Applications," *Journal of the Aeronautical Sciences*, Vol. 81, Pt. 3, 1941, pp. 104–108.

[9]Gakhov, F. D., *Boundary Value Problems*, Pergammon, New York, 1966.

[10]Milne-Thomson, L. M., *Theoretical Hydrodynamics*, MacMillan, London, 1968.

Use of the Kirchhoff Method in Acoustics

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Introduction

SOUND can be considered as the propagating part of the full solution of some fluid-flow problems. In a few cases, the radiated sound can be found this way. However, in most cases of practical interest one cannot even numerically find a full solution everywhere in the flowfield because of diffusion and dispersion errors due to increasing mesh size in the far field. Thus, it is often advantageous, or even necessary, to develop ways of finding the far-field noise from near-field solutions.

The various approaches used to find the far-field noise can be classified as follows:

1) *Full Flowfield Solution:* Calculation of the full nonlinear flowfield including far-field waves. Practical grid densities are generally insufficient to numerically resolve the details of the acoustic three-dimensional far field.

2) *Acoustic Analogy:* Nonlinear near-field solution plus Ffowcs Williams and Hawkings equation. We should note that there are substantial difficulties in including the nonlinear quadrupole term in the volume integrals.

3) *Nonlinear Near-Field Plus Kirchhoff Method:* Calculation of the nonlinear near and midfield with the far-field solutions found from a linear Kirchhoff formulation evaluated on a surface surrounding the nonlinear field. The full nonlinear equations are solved in the first region (near field), usually numerically, and a surface integral of the solution over a control surface gives enough information for the analytical calculation in the second region (far field). The method has the advantage of including the full diffraction and focusing effects and eliminates the propagation of the reactive near field. This technique is the basis of this technical note.

Kirchhoff's formula was first published in 1882. Morgans[1] derived a Kirchhoff formula for a moving surface. Hawkings[2] and Morino[3,4] rederived that formula for some special cases. Recently, Farassat and Myers[5] rederived the general Kirchhoff formula using generalized derivatives. The authors also used the method[6,7] for calculation of far-field noise due to blade-vortex interactions (BVI).

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Formulation

A Green's function approach was used to derive the Kirchhoff formula in a coordinate system fixed to the airfoil which moves with a freestream velocity. Our approach is similar to that of Morino.[3,4] The Green's function approach is relatively simple and also the coordinate system used is suitable for BVI calculations. A full three-dimensional formulation is used, because the Green's function is simpler in this case, and because the method can be easily extended to include spanwise variations to model three-dimensional BVI.

If we assume that all of the acoustic sources are enclosed by an imaginary closed surface S, then we can prove using Green's theorem,[6,7] that the pressure distribution outside a rigid fixed surface $(\partial S/\partial t_1)$ is

$$p(x_o,y_o,z_o) = -\frac{1}{4\pi}\int_S\left[\frac{1}{r_o}\frac{\partial p}{\partial n_o} + \frac{1}{c_o r_o \beta^2}\frac{\partial p}{\partial t}\left(\frac{\partial r_o}{\partial n_o} - M\frac{\partial x_o'}{\partial n_o}\right) + \frac{p}{r_o^2}\frac{\partial r_o}{\partial n_o}\right]_\tau \mathrm{d}S_o' \tag{1}$$

where

$$r_o = \left\{(x-x')^2 + \beta^2\left[(y-y')^2 + (z-z')^2\right]\right\}^{1/2}$$

$$\tau = \frac{[r_o - M(x-x')]}{c_o\beta^2}$$

$$\beta = (1-M^2)^{1/2}$$

where "$'$" denotes a point on the Kirchhoff surface element $\mathrm{d}S_o'$ and subscript "o" denotes the transformed variables, using the well-known Prandtl-Glauert transformation:

$$x_o = x, \qquad y_o = \beta y \qquad z_o = \beta z$$

$n = (n_x, n_y, n_z)$ is the outward normal to the surface S, and subscript τ implies the evaluation at the retarded time $t_1 = t - \tau$.

Also using the Green's function approach, results can be obtained for a rotating Kirchhoff surface that might be useful for helicopter rotor calculations. The values of the pressure coefficient c_p and its normal derivatives on an arbitrary surface around an arbitrary flow are enough to give the far-field radiation at any arbitrary external point. In this work, we have chosen to use a rectangular box for the surface S.

Since Kirchhoff's method assumes that linear equations hold outside this control surface S, it must be chosen large enough to include the region of significant nonlinear behavior. However, due to increasing mesh spacing, the accuracy of the numerical solution is limited to the region immediately surrounding the moving blade. As a result, S cannot be so large as to lose accuracy in the numerical solution of the midfield. Thus, a judicious choice of S is required for the effectiveness of the Kirchhoff method in the case of nonlinear waves.

A half-period of a linear spherical sine wave is used as a test case for the method.

$$c_p(r,t) = \frac{\sin\left[\dfrac{(t-\tau)\pi}{A}\right]}{r_\beta} \qquad \text{for } 0<t-\tau<A$$

$$c_p = 0 \qquad \text{elsewhere} \tag{2}$$

where A is the wave thickness (or half-period). The rectangular box-shaped control surface is shown in Fig. 1.

Results and Discussion

Figure 2 shows results for a freestream Mach number $M = 0.8$ at a point $(x, y, z) = (-0.5, 0, 0)$ for a reasonable rectangular-shaped control surface: $x_s = \pm 0.30$, $y_s = \pm 0.30$, (x_s, y_s are the coordinates of the control surface), 60×60 mesh points, mesh size $\Delta x = \Delta y = 0.01$, span = 4, 100 strips in the

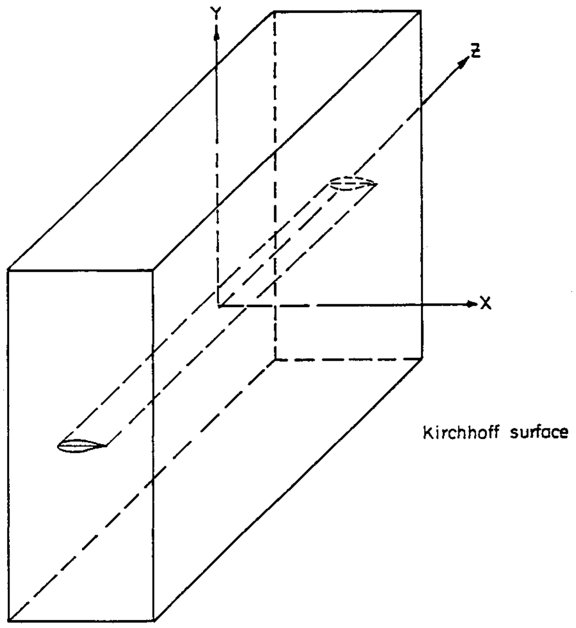

Fig. 1 Rectangular Kirchhoff surface.

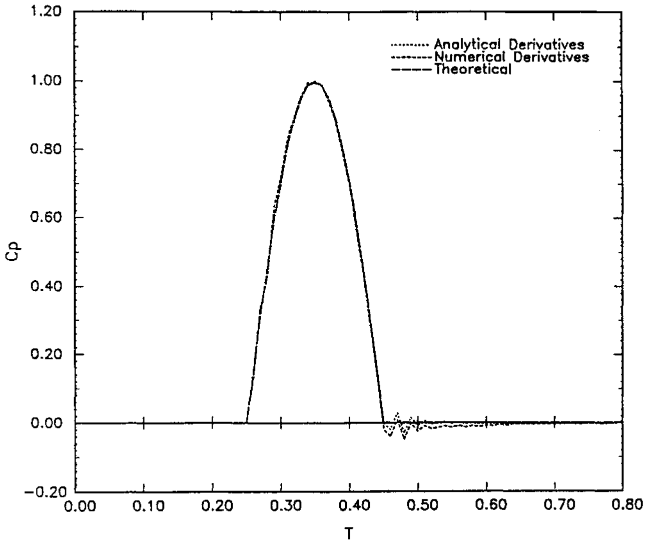

Fig. 2 Normalized spherical wave solution at (−0.5, 0, 0), $M = 0.80$, Kirchhoff surface at (−0.3, 0.3) with 60 × 60 mesh points, $\Delta x = \Delta y = 0.01$, span = 4, 100 strips in the z direction, $A = 0.2$.

z direction, sound velocity $c_o = 10$ (arbitrary units), and wave thickness $A = 0.2$. The first curve shows the theoretical results, the second curve shows results with analytical evaluation of the time and normal derivatives, and the third one shows results with numerical evaluation of the derivatives. The agreement is very good. The analytical derivatives produce a little better results, as expected. Even for a Mach number $M = 0.99$, the results were within 1% of the theoretical solution, with good wave shape.

The same results were calculated for a coarse mesh (6×6 mesh points, $\Delta x = \Delta y = 0.1$). The numerical calculations underpredict the peak of the wave about 5%, which is still satisfactory for such a coarse mesh. In the z direction, we used 100 strips for a span = 4, which proved to be enough. Also, tip end surfaces were found to have a small effect (less than 1%) and can be excluded from the calculations, for distances up to 20 chordlengths and span not less than 4. This is an important result, because the inclusion of the tip surfaces requires extensive memory for storing two-dimensional (BVI) results which are needed to compute the tip-surface configuration.

Also, the wave thickness was varied. We noticed that the thicker wave produces more accurate results as expected. It is

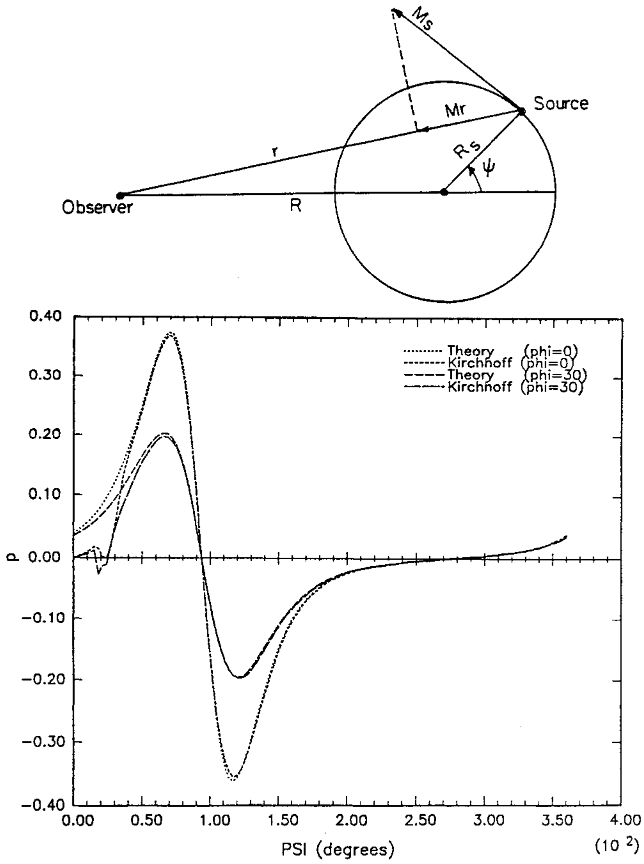

Fig. 3 a) Rotating source geometry; and b) comparison between theoretical and Kirchhoff solution for a rotating source, for two points ($\phi = 0$ deg and $\phi = 30$ deg).

also possible to match the results from the Kirchhoff method to the results from an aerodynamic near-field code as shown in Refs. 6 and 7. Finally, we examined the same effects for a more general surface corresponding to a C mesh as is usually used in the numerical solution of the Euler equations with satisfactory results.

High-speed compressibility helicopter noise can be also treated using the Kirchhoff formulation. The source can be decomposed in monopole, dipole, and quadrupole contributions. If we consider only the monopole contribution of a single rotating source, we have (linear thickness impulsive noise)

$$p = \frac{\partial}{\partial t}\left[\frac{K}{r\,|\,1 - M_r\,|}\right]_\tau \tag{3}$$

where r and M_r are defined in Fig. 3a, K is a constant (source strength/4π), the subscript τ denotes the evaluation at the retarded time, and $1/\,|\,1-M_r\,|$ is the Doppler amplification factor. The solution can be found analytically for an arbitrary point from Eq. (3). We can also use a Kirchhoff surface around the source. Equation (3) is used to calculate the solution around a cylindrical surface (radius = R_k) that encloses the source, then Eq. (2) is used.

Figure 3b shows the comparison between the theoretical and the Kirchhoff solution for a rotating source (hover): $R_s = 24$ ft (source radius); $M_s = 0.7$ (source Mach number); $R_k = 1.2 \times R_s$ (radius of Kirchhoff surface); $z_s = 480$ ft (height of Kirchhoff surface); $K = 1$; $c_o = 1120$ ft/s, 120 points in the z direction; $r = 536.7$ ft (position of the observer); 3-deg increments around the surface were used (120 points). Central differences were used for both space and time derivatives. The tips of the Kirchhoff cylinder surface (bases) were found to have no important contribution. Two cases are shown as follows:

1) The observer is at the rotor plane, $\phi = 0$ deg, $(x, y, z) = (-536.7, 0, 0)$.

2) The observer is at the same distance, but at an angle $\phi = 30$ deg from the rotor plane, $(x, y, z) = (-480, 0, 240)$.

The agreement between theoretical and Kirchhoff solutions is very good for both cases with error less than 1% for peak values. The discrepancy at the beginning is due to the sudden jump at the solution there. Case 2 has a lower signal, as expected, since the thickness noise is maximum at the rotor plane.

This simple example shows the viability of the method for three-dimensional helicopter noise calculations. The Kirchhoff method is not limited to the sonic cylinder, and the observer can be out of the rotor plane. Solutions can also be obtained for a moving rotor. This powerful method can be utilized for the development of a set of simple portable Kirchhoff subroutines for the calculation of the far-field noise using the input given from any aerodynamic near/midfield code.

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References

[1]Morgans, R. P., "The Kirchhoff Formula Extended to a Moving Surface," *Philosophical Magazine,* Vol. 9 (s.7, No. 55), 1930, pp. 141-161.

[2]Hawkings, D. L., "Noise Generation by Transonic Open Rotors," Westland Research Paper 599, July 1979.

[3]Morino, L., "A General Theory of Unsteady Compressible Aerodynamics," NASA CR 2646, Dec. 1974.

[4]Morino, L., "Steady, Oscillatory, and Unsteady Subsonic and Supersonic Aerodynamics—Production Version (Soussa-p 1.1) Vol. I—Theoretical Manual," NASA CR 159130, Jan. 1980.

[5]Farassat, F. and Myers, M. K., "Extension of Kirchhoff's Formula to Radiation from Moving Surfaces," *Journal of Sound and Vibration,* Vol. 123, No. 3, June 1988, pp. 451-460.

[6]George, A. R. and Lyrintzis, A. S., "Acoustics of Transonic Blade-Vortex Interactions," *AIAA Journal,* Vol. 26, July 1988, pp. 769-776.

[7]Lyrintzis, A. S. and George, A. R., "Transonic Blade-Vortex Interactions: The Far Field," *Proceedings of the American Helicopter Society National Specialists' Meeting on Aerodynamics and Aeroacoustics,* Arlington, Texas, American Helicopter Society, Alexandria, VA, Feb. 1987.

New Series Expansion Method for the Solution of the Falkner-Skan Equation

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Introduction

THE Falkner-Skan equation is a celebrated equation in fluid mechanics. Its solutions are well known as similar solutions. The study of the similar solutions in the past has contributed greatly to our understanding of the mechanism which governs the process of heat, mass, and momentum transfer through laminar boundary layers. It is of interest to note that the similar solutions are also required in several series solutions of partial differential equations, where they usually appear as the first term. Similar solutions are important because they also serve as the foundation for a number of more general methods for boundary-layer analysis used for estimating transfer rates for both similar and nonsimilar boundary layers. Because of the theoretical and practical importance of the Falkner-Skan equation as stated above and because no closed-form solution is known, a number of numerical solutions for many discrete values of β have been tabulated and many scientists have worked on it, such as Cebeci and Keller,[1] Smith,[2] and Evans,[3] among others. However, all of their solutions are in discrete form. Recently, Aziz and Na[4] described a new approach for this problem. They explored a regular perturbation expansion for f as a power series in β for this problem. Compared to the previous methods, the power-series method is similar both in conception and computation. But the resulting series is found to converge only for very small values of β. To improve the range of applicability and accuracy, the Shanks transformation is applied repeatedly. This process is not only inconvenient but also inaccurate. Under the motivation of this approach and the research of Van Dyke,[5,6] we derive a new series expansion method for this problem. Four series for $f''(0)$ and four series for δ_1 originating at $\beta_0 = 2, 1, 0.3, 0$, respectively, have been worked out. To overcome the difficulty near separation, a common term is constructed and used to estimate the remainder of the series originating at $\beta_0 = 0$. The surprising agreement of the results with the well-known numerical solutions shows the high accuracy of the present method.

Mathematical Model

The celebrated Falkner-Skan equation in fluid mechanics is of the form

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1 \tag{2}$$

Substituting $t = \beta_0 - \beta$ into Eq. (1), we have

$$f''' + ff'' + \beta_0(1 - f'^2) = t(1 - f'^2) \tag{3}$$

Assuming

$$f = \sum_{n=0}^{\infty} t^n f_n \tag{4}$$

we have

$$f_0''' + f_0 f_0'' + \beta_0(1 - f_0'^2) = 0 \tag{5}$$

$$f_0(0) = f_0'(0) = 0, \qquad f_0'(\infty) = 1 \tag{6}$$

$$f_n''' + f_0 f_n'' - 2\beta_0 f_0' f_n' + f_0'' f_n = \delta_1 n - \sum_{r=1}^{n} f_{r-1}' f_{n-r}' - \sum_{r=1}^{n-1} f_r f_{n-r}'' + \beta_0 \sum_{r=1}^{n-1} f_r' f_{n-r}' \tag{7}$$

$$f_n(0) = f_n'(0) = 0, \qquad f_n'(\infty) = 0 \tag{8}$$

where $n = 1, 2, 3, \ldots$ and δ_{1n} is the well-known Kronecker delta.

The skin-friction factor is

$$f''(0) = \sum_{n=0}^{\infty} f_n''(0) t^n = \sum_{n=0}^{\infty} a_n t^n \tag{9}$$

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